E9 203 Compressed Sensing & Sparse SP

07 May 2021

Last time:
- Analysis of thresholding algos. via RIC.

Today:
- Proof of Thm. 6.18

Thm. 6.18 Suppose the $3s^{th}$ RIC of $A \in \mathbb{C}^{m \times N}$ satisfies $\delta_{3s} < \frac{1}{\sqrt{3}} \simeq 0.5773$. Then, for $x \in \mathbb{C}^N$, $e \in \mathbb{C}^m$, $S \subset [N]$ with $|S| = s$, the seq. x^n def. by

IHT : $x^{n+1} = \underline{\underline{H_s}}\left(\underline{x^n + A^H(y - Ax^n)}\right)$

and

HTP$_1$: $S^{n+1} = L_s\left(\underline{x^n + A^H(y - Ax^n)}\right)$

HTP$_2$: $x^{n+1} = \arg\min \{ \|y - Az\|_2 ,\ \mathrm{supp}(z) \subset S^{n+1} \}$

with $y = Ax + e$ satisfies, for any $n \geq 0$,

$$\boxed{\|x^n - x_S\|_2 \leq \rho^n \|x^0 - x_S\|_2 + \tau \|A x_{\overline{S}} + e\|_2}$$

where

$\rho = \sqrt{3}\,\delta_{3s} < 1$, $\tau \leq \frac{2.18}{1-\rho}$ for IHT

$\rho = \sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}} < 1$, $\tau \leq \frac{5.15}{1-\rho}$ for HTP.

Lemma 6.26 Given $e \in \mathbb{C}^m$, $S \subset [N]$, $|S| \leq s$,

$$\boxed{\|(A^H e)_S\|_2 \leq \sqrt{1+\delta_s}\, \|e\|_2}$$

Proof: $\|(A^H e)_S\|_2^2 = \langle A^H e, (A^H e)_S \rangle$
$= \langle e, A\underline{\underline{(A^H e)_S}} \rangle$
$\leq \|e\|_2 \cdot \|A\underline{(A^H e)_S}\|_2$
$\leq \|e\|_2 \cdot \sqrt{1+\delta_s}\, \|\underline{(A^H e)_S}\|_2$. ☒

Proof of Thm. 6.18

Want t.s.t., for any $n \geq 0$

$$\boxed{\|x^{n+1} - x_S\|_2 \leq \rho \|x^n - x_S\|_2 + (1-\rho)\tau \|A x_{\overline{S}} + e\|_2}$$

If true, by induction,

$\leq \rho^2 \|x^{n-1} - x_S\|_2 + (1+\rho)(1-\rho)\tau \|A x_{\overline{S}} + e\|_2$

$\leq \rho^{n+1} \|x^0 - x_S\|_2 + (\underbrace{1+\rho+\dots+\rho^n}_{\frac{1-\rho^{n+1}}{1-\rho} < \frac{1}{1-\rho}})(1-\rho)\tau \|A x_{\overline{S}} + e\|_2$.

For both IHT & HTP, $S^{n+1} \triangleq \mathrm{supp}(x^{n+1})$ is the idx set corresp. the s largest entries of $x^n + A^H(y - Ax^n)$,

$\Rightarrow \|(\underbrace{x^n + A^H(y - Ax^n)}_{v})_S\|_2^2 \leq \|(x^n + A^H(y - Ax^n))_{S^{n+1}}\|_2^2$

$\|v_S\|_2^2 = \underline{\|v_{S \cap S^{n+1}}\|_2^2} + \|v_{S \setminus S^{n+1}}\|_2^2$

$\|v_{S^{n+1}}\|_2^2 = \underline{\|v_{S \cap S^{n+1}}\|_2^2} + \|v_{S^{n+1} \setminus S}\|_2^2$

$\Rightarrow \|(x^n + A^H(y - Ax^n))_{S \setminus S^{n+1}}\|_2 \leq \|(x^n + A^H(y - Ax^n))_{S^{n+1} \setminus S}\|_2$

LHS $= \|(x_S - x^{n+1} + x^n - x_S + A^H(y - Ax^n))_{S \setminus S^{n+1}}\|_2$

$\geq \|(x_S - x^{n+1})_{S \setminus S^{n+1}}\|_2 - \|(\underline{x^n - x_S + A^H(y - Ax^n)})_{\underline{\underline{S \setminus S^{n+1}}}}\|_2$

RHS $= \|(\underline{x^n - x_S + A^H(y - Ax^n)})_{\underline{\underline{S^{n+1} \setminus S}}}\|_2$

Let $\underline{S \Delta S^{n+1}} \triangleq (S \setminus S^{n+1}) \cup (S^{n+1} \setminus S)$ [symm. diff. betw. S & S^{n+1}]

We get

$\|(x_S - x^{n+1})_{S \setminus S^{n+1}}\|_2 \leq \|(x^n - x_S + A^H(y - Ax^n))_{\underline{\underline{S \setminus S^{n+1}}}}\|_2$
$+ \|(x^n - x_S + A^H(y - Ax^n))_{\underline{\underline{S^{n+1} \setminus S}}}\|_2$

$$\boxed{\|(x_S - x^{n+1})_{S \setminus S^{n+1}}\|_2 \leq \sqrt{2}\, \|(x^n - x_S + A^H(y - Ax^n))_{S \Delta S^{n+1}}\|_2} \quad (*)$$

IHT: $x^{n+1} = (x^n + A^H(y - Ax^n))_{S^{n+1}}$

Then, $\|x^{n+1} - x_S\|_2^2 = \|(x^{n+1} - x_S)_{S^{n+1}}\|_2^2 + \|(x^{n+1} - x_S)_{\overline{S^{n+1}}}\|_2^2$

$= \|(x^n - x_S + A^H(y - Ax^n))_{S^{n+1}}\|_2^2 + \underline{\|(x^{n+1} - x_S)_{S \setminus S^{n+1}}\|_2^2}$

$\leq \|(\quad " \quad)_{S^{n+1}}\|_2^2$

$+\ 2\|(\quad " \quad)_{S \Delta S^{n+1}}\|_2^2$

$\leq 3\|(\quad " \quad)_{S \cup S^{n+1}}\|_2^2$.

Write $y = Ax + e = \underline{A x_S + e'}$, $e' \triangleq \underline{\underline{A x_{\overline{S}} + e}}$.

Note: $|\underset{\sim}{S} \cup \underset{\sim}{S^{n+1}} \cup \underset{\sim}{\mathrm{supp}(x^n - x_S)}| \leq 3s$, use

Lemma 6.16: $\|((I - A^H A)v)_S\|_2 \leq \delta_t \|v\|_2$ if $|S \cup \mathrm{supp}(v)| \leq t$

Lemma 6.20: $\|(A^H e)_S\|_2 \leq \sqrt{1+\delta_s}\, \|e\|_2$ if $|S| \leq s$.

$\|x^{n+1} - x_S\|_2 \leq \sqrt{3}\, \|(x^n - x_S + A^H(y - Ax^n))_{S \cup S^{n+1}}\|_2$

$= \sqrt{3}\, \|(x^n - x_S + A^H A(x_S - x^n) + A^H e')_{S \cup S^{n+1}}\|_2$

$\leq \sqrt{3}\left[\|((I - A^H A)(x^n - x_S))_{S \cup S^{n+1}}\|_2 + \|(A^H e')_{S \cup S^{n+1}}\|_2\right]$

$\leq \sqrt{3}\left[\delta_{3s} \|x^n - x_S\|_2 + \sqrt{1+\delta_{2s}}\, \|e'\|_2\right]$

Thus,

$\|x^{n+1} - x_S\|_2 \leq \rho \|x^n - x_S\|_2 + (1-\rho)\tau \|A x_{\overline{S}} + e\|_2$

with $\rho \triangleq \sqrt{3}\,\delta_{3s} < 1$ if $\delta_{3s} < \frac{1}{\sqrt{3}}$, $(1-\rho)\tau = \sqrt{3}\sqrt{1+\delta_{2s}}$
$\leq \sqrt{3}\sqrt{1 + 1/\sqrt{3}}$
$= \sqrt{3+\sqrt{3}} \leq 2.18$.

HTP: $x^{n+1} = \arg\min \{ \|y - Az\|_2 ,\ \mathrm{supp}(z) \subset S^{n+1} \}$

$\Rightarrow x^{n+1}$ is the best ℓ_2 approx to y in $\{Az, \mathrm{supp}(z) \subset S^{n+1}\}$

$\Rightarrow \langle y - Ax^{n+1}, Az \rangle = 0$ whenever $\mathrm{supp}(z) \subset S^{n+1}$

$\Rightarrow \langle A^H(y - Ax^{n+1}), \underline{z} \rangle = 0$ " "

$\Rightarrow (A^H(y - Ax^{n+1}))_{S^{n+1}} = \underline{0}$.

From this and $\circledast$

$$
\begin{aligned}
\|x^{n+1}-x_S\|_2^2 &= \|(x^{n+1}-x_S)_{S^{n+1}}\|_2^2 + \|(x^{n+1}-x_S)_{S\setminus S^{n+1}}\|_2^2 \\
&\leq \|(x^{n+1}-x_S+A^H(y-Ax^{n+1}))_{S^{n+1}}\|_2^2 \\
&\qquad + 2\|(x^n-x_S+A^H(y-Ax^n))_{S\Delta S^{n+1}}\|_2^2 \\
&\leq \left[\|((I-A^HA)(x^{n+1}-x_S))_{S^{n+1}}\|_2 + \|(A^He')_{S^{n+1}}\|_2\right]^2 \\
&\quad + 2\left[\|((I-A^HA)(x^n-x_S))_{S\Delta S^{n+1}}\|_2 + \|(A^He')_{S\Delta S^{n+1}}\|_2\right]^2
\end{aligned}
$$

Again use Lemma 6.16 & 6.20:

$$
\begin{aligned}
\Rightarrow \|x^{n+1}-x_S\|_2^2 &\leq \left[\delta_{2s}\|x^{n+1}-x_S\|_2 + \sqrt{1+\delta_s}\,\|e'\|_2\right]^2 \\
&\quad + 2\left[\delta_{3s}\|x^n-x_S\|_2 + \sqrt{1+\delta_{2s}}\,\|e'\|_2\right]^2
\end{aligned}
$$

Rearranging,

$$
\begin{aligned}
2\left[\delta_{3s}\|x^n-x_S\|_2 + \sqrt{1+\delta_{2s}}\,\|e'\|_2\right]^2
&\geq (1-\delta_{2s}^2)\left(\underline{\|x^{n+1}-x_S\|_2 + \frac{\sqrt{1+\delta_s}}{1+\delta_{2s}}\|e'\|_2}\right)\underbrace{\left(\|x^{n+1}-x_S\|_2 - \frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\|e'\|_2\right)}_{\geq 0 \text{ WLOG}} \\
&\geq (1-\delta_{2s}^2)\left(\underline{\|x^{n+1}-x_S\|_2} - \frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\|e'\|_2\right)^2
\end{aligned}
$$

Take square root & rearrange

$$
\|x^{n+1}-x_S\|_2 \leq \frac{\sqrt{2}\,\delta_{3s}}{\sqrt{1-\delta_{2s}^2}}\|x^n-x_S\|_2 + \left(\frac{\sqrt{2}}{\sqrt{1-\delta_{2s}}} + \frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\right)\|e'\|_2
$$

Which is in the form we want, with

$$
\rho \triangleq \frac{\sqrt{2}\,\delta_{3s}}{\sqrt{1-\delta_{2s}^2}} \leq \frac{\sqrt{2}\,\delta_{3s}}{\sqrt{1-\delta_{3s}^2}} < 1 \text{ if } \delta_{3s} < \frac{1}{\sqrt{3}}
$$

and $(1-\rho)\tau = \dfrac{\sqrt{2}}{\sqrt{1-\delta_{2s}}} + \dfrac{\sqrt{1+\delta_s}}{1-\delta_{2s}} \leq 5.15$ if $\delta_{3s} < \dfrac{1}{\sqrt{3}}$. $\boxtimes$